Section-D

Laplace Transform (LT) and Z-Transform

Introduction

- Introduced by Pierre Simon Marquis de Laplace (1749-1827). Practically used by Oliver Heaviside (1850-1925)
- Transform relationship between x(t) and X(s):

Laplace Transform

$$x(t) \rightleftarrows X(s)$$

Inverse Laplace Transform

Laplace Transform Pairs

- The Laplace transform originated from the Fourier transform.
- Definition:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}\,dt$$

$$x(t) = \frac{1}{2\pi j}\int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st}\,ds$$

where s is a complex variable that is defined as $s=\sigma+jw$.

Laplace transform

•Definition:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \frac{1}{2\pi j} \int_{-\sigma-j\omega}^{\sigma+j\omega} X(s)e^{+st}ds$$

• For a causal system, where t≥0, the Laplace Transform will be:

$$X(s) = \int_{0}^{\infty} x(t)e^{-st}dt$$

$$x(t) = \int_{0}^{\sigma+j\omega} X(s)e^{+st}ds$$

- The Laplace transform originated from the Fourier transform.

$$X(s)\big|_{s=\sigma+j2\pi f} = \int_0^{\infty} x(t)e^{-(\sigma+j2\pi f)t}dt$$

$$X(\sigma + j2\pi f) = \int_0^{\infty} \left[x(t)e^{-\sigma t}\right]e^{-j2\pi ft}dt$$

$$\int_0^{\infty} y(t)\ e^{-j2\pi ft}\ dt$$

Examples of Laplace Transform

• Example 1:

A signal is defined as

$x(t) = 1 \quad t \geq 0$
$\quad\;\; = 0 \quad \text{elsewhere}$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt = \int_0^{\infty} (1) e^{-st} dt$$

$$= \frac{1}{s}$$

•Example 2:

A signal is defined as

$$x(t) = e^{-at} \qquad t > 0$$
$$= 0 \qquad t < 0$$

$$X(s) = \int_0^\infty x(t) e^{-st} dt = \int_0^\infty \exp(-at) e^{-st} dt$$

$$= \int_0^\infty e^{-(a+s)t} dt = -\left. \frac{e^{-(a+s)t}}{(a+s)} \right|_0^\infty$$

$$= -\frac{e^{-(a+s)\infty}}{(a+s)} + \frac{\exp^{-(a+s)0}}{(a+s)} = \frac{1}{(s+a)}$$

• <u>Example 3</u>

A sinusoidal signal is expressed as

$$x(t) = \cos(2\pi f_1 t) \qquad t \geq 0$$

$$= 0 \qquad t < 0$$

$$X(s) = \int_0^{\infty} x(t) e^{-st} dt = \int_0^{\infty} \cos(2\pi f_1 t) e^{-st} dt$$

By defining the cosine function in terms of complex exponentials, the Laplace transform is

$$X(s) = \frac{s}{\left(s^2 + (2\pi f_1)^2\right)}$$